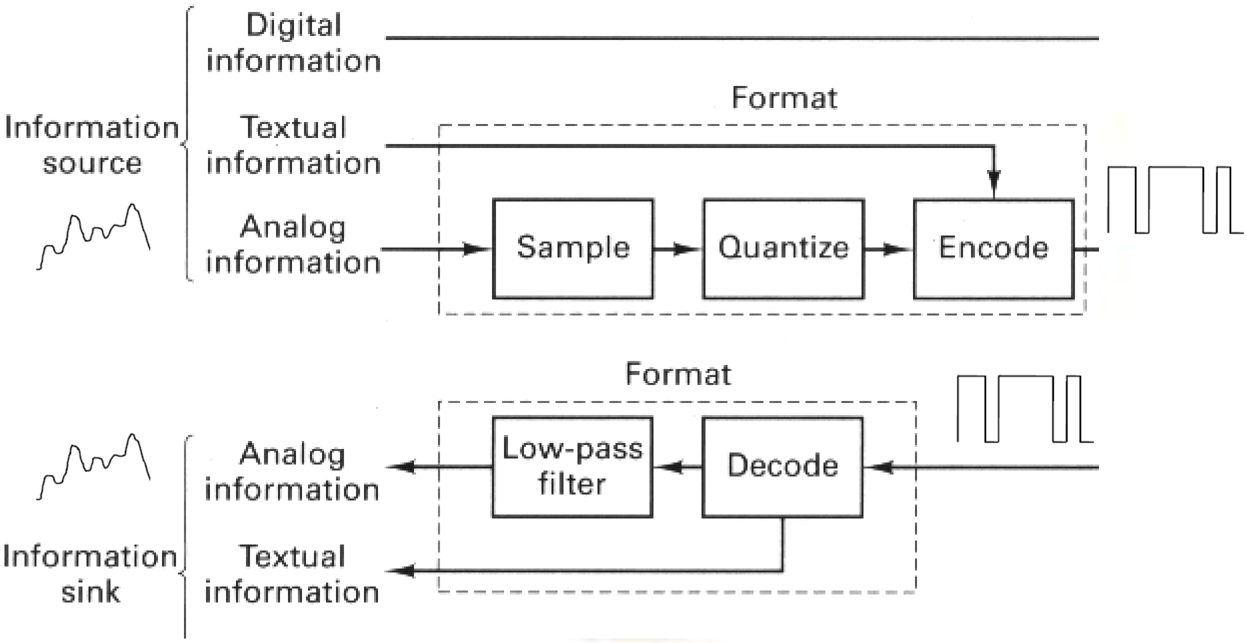
The sampling theorm

The link between the an analoge waveform and its sampled version is provided by what is known as the sampling process. This process can be implemented is several ways, the most popular being the sample-and-hold operation. In this operation, a switch and strong mechanism form a sequence of samples of the continous input waveform. The output of the sampling process is called pulse amplitude modulation (PAM) because the successive output intervals can be described as a sequence pulses with amplitudes derived from the input waveform samples. The analoge wave form can be approximately retrieved from a PAM waveform by simple low-pass filtering. The restriction, stated in terms of the sampling rate $f_s = \frac{1}{T_s}$, (where T_s is the sampling period) is known as the Nyquest criterion and can be expressed as

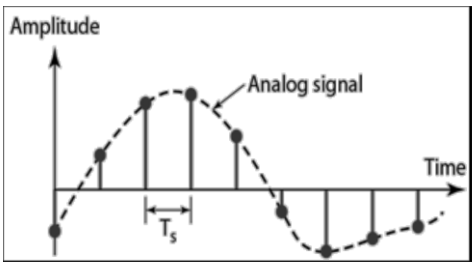
$$f_s \geq 2f_m$$

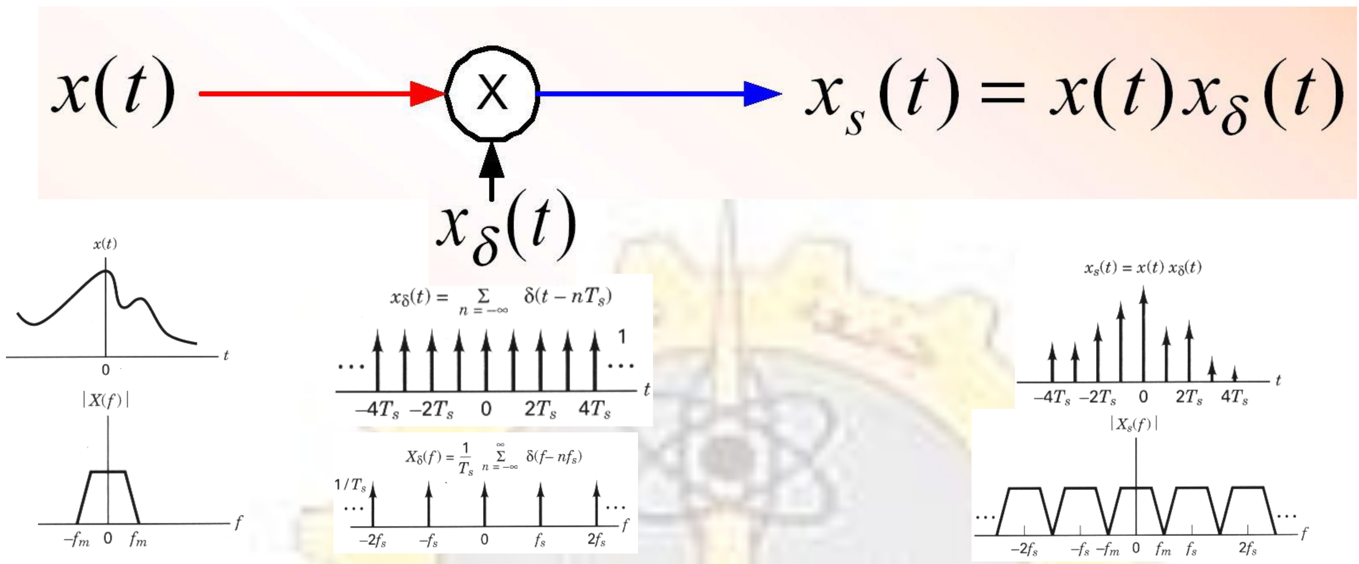
The sampling rate $f_s = 2f_m$ is also called the Nyquest rate.



Impulse samlong

The sampling theorm can be validation by using the frequency convolution properity of the Fourier transmission. Let x(t) is the input signal





The out put $x_s(t)$, which is the sampled version of $x(t)$, in time domain can be written as

$$x_s(t) = x(t)\delta(t)$$

Using the sifiting property of the impulse function, $x_s(t)$ can be rewritten as

$$x_s(t) = \sum_{n=-\infty}^{\infty} x(t)\delta(t - nT_s) = \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t - nT_s)$$

Using the frequency convolution properity of the Fourier transform, the time domain product $x(t)\delta(t)$ can be transformed to frequency domain convolution $X(f)X_\delta(f)$

$$x_s(f) = \frac{1}{T_s}\sum_{n=-\infty}^{\infty} X(f - nf_s)$$

Example:

Suppose that an analog signal is given as $x(t) = 5\cos(2\pi \times 10^3 t)$ and is sampled at the rate 8kHz. Sketch the spectrum for the original signal and the sampled signal from 0 to 20kHz.

Signal reconstruction (Interpolation)

The analog signal (origonal signal) can be recovered from its sampled version by two simplified steps. First, the digitally processed data $y(n)$ are converted to igeal impulse train $y_s(t)$, in which each has amplitude propertional to digital output $y(n)$, and two conseqecutive impulses are seperated by a sampling period of T; second, the analog reconstruction filter is applied to the ideally recovered sampled signal $y_s(t)$ to obtain the recovetred analog signal. There are three cases for recovery of the original signal spectrum $X(f)$;

Case 1: $f_s = 2f_m$

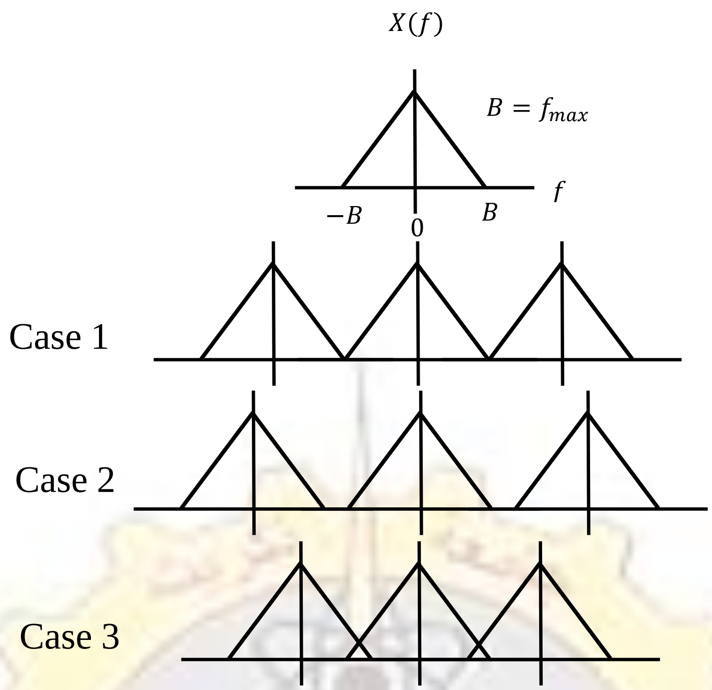
In thhis case the Nyquist frequency is equal to the macximum frequency of the analog signal $x(t)$, an ideal lowpass reconstructure filter is required to recover the analog signal spectrum. This is an impractical case.

Case 2: $f_s > 2f_m$

In this case, there is a separation betweenthe highest frequency edage of the baseband spectrum and the lowest edge of the first replica. Therefore, a practical lowpass reconstructure filter can be designed to reject all the images and achieve the original signal spectrum.

Case 3: $f_s < 2f_m$

This case violates the condition of the Shannon sampling theorm. Spectral overlapping occurs between the original baseband spectral and the spectrum of the first replica. Even when using an ideal lowpass filter to recover the original spectrum, there are still some foldover frequency components from the adjacent replica. This is aliasing, where the recover baseband spectrum suffered spectral distortion.



Examlpe:

Assume that an analog signal is given by

$a)\ x(t) = 5\cos(4 \times 10^3 \pi t) + 3\cos(6 \times 10^3 \pi t),\ t \geq 0$

$b)\ x(t) = 5\cos(4 \times 10^3 \pi t) + 1\cos(10 \times 10^3 \pi t),\ t \geq 0$

and is sampled at the rate of 8kHz. Sketch the spectrum for the sampled signal from 0 to 20kHz. Sketch the recovered analoge signal spectrum if an ideal LPF with a cutoff frequency of 4KHz used to filter the sampled signal ($y(n) = x(n)$)to recover the original signal.

Quantization

One of the basic choices in quantization is the number of discrete quantization levels to use. The fundamental tradeoff in this choice is the resulting signal quality versus the amount of data needed to represent each sample.

Classification of Quantization

There are two types of quantization - Uniform Quantization and Non-uniform Quantization. The type of quantization in which the quantization levels are uniformly spaced is termed as a Uniform Quantization. The type of quantization in which the quantization levels are unequal and mostly the relation between them is logarithmic, is termed as a Non-uniform Quantization.

Uniform quantization: divide interval from $+X_{max}$ to $-X_{max}$ into 2^m intervals of length $\Delta=(2X_{max}/2^m)$ for a m-bit quantizer

Unipolar quantizer: deals with analog signals ranging from 0 volt to a positive reference voltage

Polar quantizer: deals with analog signals ranging from a negative reference to a positive reference voltage.

$$\Delta = \frac{X_{max} - X_{min}}{L} \qquad L = 2^m$$

The quantization level x_q corresponding to the binary code and can be written as

$$x_q = X_{min} + i\Delta \qquad i = 0,1,\cdots,L-1$$

For any system, during its functioning, there is always a difference in the values of its input and output. The processing of the system results in an error, which is the difference of those values. The difference between an input value and its quantized value is called a Quantization Error.

When the DAC output the analog amplitude x_q with finite precision, it introduces quantization error defined as $e_q = x_q - x$

Example

Draw the quantization level and the error as a function of input signal x, if m=3 for unipolar and polar quatizers.

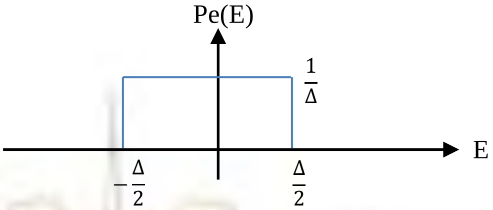
Example

Assuming that a 3-bit ADC channel accepts analog input ranging from 0 to 5volts, determine

a- The number of quatization levels
b- The step size of the quantizer or resolution.
c- The quantization level when the analog voltage is 3.2 volts.
d- The binary code produced by the ADC.
e- The quatization error when the analog voltage is 3.2 volts.

Quantization error and signal to noise ratio SNR.

Quantization error is uniformly distribution. The probability density function PDF of quantization error as



Based on the theory of probability and random variables, the power of quantization noise is related to the quantization step and given by

$$\sigma_{noise}{}^2 = E\left(e_q{}^2\right) = \int_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} \frac{1}{\Delta} e_q{}^2 de_q = \frac{\Delta^2}{12}$$

Where E() is the expectation operation. The ratio of signal power to quantization noise power (SNR) can be expressed as

$$SNR = \frac{E(x^2)}{E(e_q{}^2)}$$

If we express the SNR in term of decibels (dB), we have

$$SNR_{dB} = 10 \log_{10} SNR \qquad dB$$

$$SNR_{dB} = 10.79 + \log_{10}\left(\frac{x_{rms}}{\Delta}\right)$$

Where x_{rms} is the RMS value of the signal to be quantized x and $E(x^2) = x_{rms}^2$. Practically, the SNR can be calculated using the following foemula

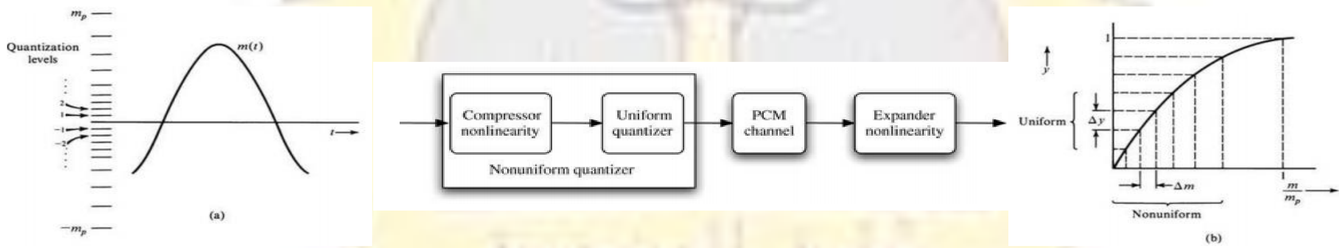
$$SNR = \frac{\frac{1}{N}\sum_{n=0}^{N-1} x^2(n)}{\frac{1}{N}\sum_{n=0}^{N-1} e_q^2(n)} = \frac{\sum_{n=0}^{N-1} x^2(n)}{\sum_{n=0}^{N-1} e_q^2(n)}$$

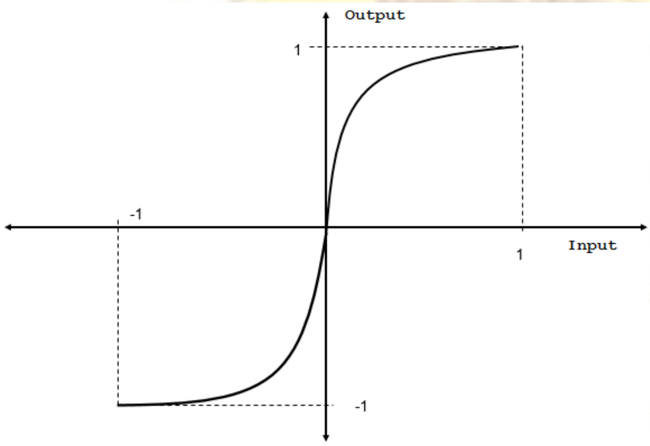
Example:

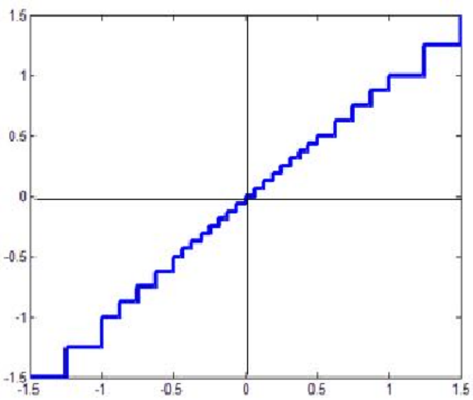
If the analog signal to be quantized is a sinusoidal waveform, that is $x(t) = Asin(2 \times 10^3 \pi t)$ and if the bipolar quantizer uses m bits, determine the SNR in term of m bits

Non-uniform quantization

One way of achieving nonuniform quantization is to use a nonuniform quantizer characteristic. More often, nonuniform quantization is achieved by first distorting the original signal with a logarithmic compression characteristic, and then using a uniform quantizer. For small magnitude signals the compression characteristic has a much steeper slope than for large magnitude signals. Thus, a given signal change at small magnitudes will carry the uniform quantizer through more steps than the same change at large magnitude. The compression characteristic effectively changes the distribution of the input signal magnitudes so that there is not a preponderance of low magnitude signals at the output of the compressor. After compression, the distorted signal is used as the input to the uniform quantizer characteristics. At the receiver, an inverse compression characteristic, called expansion, is applied so that the overall transmission is not distorted. The processing pair (compression and expansion) is usually referred to as companding.







Telephone systems use ITU standardized compression formula
μ-law: North America and Japan. For μ=255 (for 8bit codes)

$$y = sgn(x)\frac{1}{ln(1+\mu)}ln(1+\mu|x|). \qquad 0 < x < 1$$

.

A-law: Europe, rest of world

$$y = \begin{cases} sgn(x)\dfrac{A|x|}{1+ln(A)} & |x| < \dfrac{1}{A} \\ sgn(x)\dfrac{1+ln(A|x|)}{1+ln(A)} & \dfrac{1}{A} < x < 1 \end{cases}$$

The standard value is A=87.7

For both laws, the input to the compressor is

$$x = \frac{m(t)}{m_p}$$

Where $-m_p \le m(t) \le m_p$